Light Statistics

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Fields and Photons

The classical theory of electromagnetic fields provides a framework for understanding the propagation of light from a source to a sensor array such as a charge coupled device that we would use for time-dependent precision imaging, photometry, spectroscopy, or polarimetry. With it we develop tools for analyzing spatial and temporal dependence of energy flux, frequency, and polarization associated with the oscillation of the electromagnetic field [1]. Equivalently, the theories of quantum optics and photon counting statistics provide a broader foundation on which we build an understanding of discrete photons arriving at the detector some time after they are emitted, traveling at speed c, carrying energy hf, momentum hf/c and spin angular momentum $\hbar$ [2, 3]. The quantum framework provides a complete theory of their intrinsic properties as well as their interaction with matter, and their statistics. It is the preferred basis for modeling optical data over time and space. It accounts for fluctuations and correlations, applies even when only discrete single photons are measured, and explains the emission and noise of photoelectrons that is the basis of detection.

The Measurement of Individual Photons

We consider the analysis of individually counted photons. Prior to the development of the photomultiplier tube, measurement of light was done by monitoring a current of photoelectrons. Noise fluctuations in the current could be associated with changes in the power delivered by the electric field to determine intensity correlations that enabled the measurement of stellar diameters, as demonstrated profoundly in the experiments and theory developed by Hanbury Brown [4, 5]. This work led to a deeper understanding of the nature of light, and to the technology in use today in astrophysics, biology, and quantum computing [6, 7].

The precision with which physical properties of a source can be established only through a signal carried by the light it emits or scatters is limited by the randomness of the photons in the signal and by intinsic quantum uncertainty in the measurement process. Even when the source is perfectly steady, if only a few photons are measured in a given time interval the number detected will fluctuate randomly. The scale of the fluctuations is determined by photon statistics and correlation and that introduces an inherent uncertainty in any measurement that depends on light. When a telescope with a single large mirror images direction onto a photosensitive array, the smaller the ratio of the pixel size to its distance from the mirror, the smaller the angle of space subtended by the pixel. In addition to statistics of photon counting, there is a limit to that accuracy because of the quantum properties of the photon. The larger the aperture of the telescope, the less certainty in the position of the photon arriving at the entrance aperture, and the more precisely the direction to the source is determined. We can never state absolutely that the photon detected at that pixel originated in the direction the pixel geometrically represents in space. Taken over many detected photons, their intrinsic properties and the spatial limits of the detection process yield a point spread function in the image domain. It is the result of a quantum uncertainty relation in momentum and position applied to photons. It is maintained even in the limit of detecting one photon at a time, accumulating an image over seconds, minutes, hours or days. Such single-photon interference so basic to imaging in astronomy is the result of each photon taking all possible paths on its way to us. Each image we analyze is the result in a probabilistic sense of this sum of possibilities.

Images are also the result of the sum of an ensemble of photons over time, a window in which photons are accepted and sensed. You may equivalently think of it as an average over the distance in space along the path of the photons as a consequence of their speed and the duration of the detection process. An examination of the coherence of photons arriving over a time interval is limited by the duration of that interval. The longer the time, the more precisely the energy of the photons is determined, meaning that larger diffraction gratings or delay lines are required for more precise spectroscopy. Yet there is another uncertainty set by photon statistics which comes to bear on even the simplest of measurements in which both spectral and spatial bandwidth are broadened to encompass as many photons as possible. The number of photons detected is set by the flux that is collected and the time over which the integration is done. The more photons in a sample, the more precisely the average of their properties is determined.

Electronic imaging sensors for visible and near-infrared light integrate the charge produced by photoelectrons within individual pixels and convert that charge to a number that is associated with the pixel and is proportional to the number of photons that arrived during the collection interval. Thus a signal has two components: a *quantum efficiency* (q) representing the probability that the detector responds to that photon, and a digital value resulting from the successful response to a single photon which we call the *gain* (g). For n photons arriving on a detector's pixel over an integration time interval, the total digital signal is the number nqg in these digital units. For now we focus attention on the photon number itself and the statistics that govern detection at the limit of perfect quantum efficiency.

Photon Probabilities

The statistics of photons depends on the spatial and temporal character of the source and the processes that produce them. Coherent monochromatic light, such as arises from a laser, is classically a pure oscillation of the electromagnetic field that maintains constant amplitude and phase. This is a stable light source that is not in thermal equilibrium and its photons have a Poisson distribution [7, 3]. Photons from a blackbody have a thermal Gaussian distribution determined only by its temperature [8, 9]. Photons which are collectively a gas of bosons must follow a Bose-Einstein distribution [10, 11]. The connections between these idealized cases and conditions under which they apply were clarified in the early days of the development of quantum electrodynamics, and are explained well in reviews given by Pike [12, 13] that add a perspective of technology development to the Noble Prize winning development of coherence theory by Glauber [14].

In a paper considering photoelectric detection of light, Fried [15] notes that

> *It is well known that the number of photons arriving on a surface during a time interval obeys a Poisson distribution.*

He adds that photons obey Bose statistics so this is not exact. The difference between Poisson and Bose statistics for blackbody radiation is greatest on the long wavelength side of the blackbody peak. Thus if we are observing sources in visible light that are cooler than say 3000 K, the difference in the distributions is negligible. It raises a question about light from main sequence stars hotter than this, say spectral types K and earlier, but for them the differences may not be significant in practice even though Fried termed them "unreasonably" hot bodies. The surface in question would be the entrance aperture to a telescope or other instrument. While it is not self-evident that the detection through the interaction of photons with matter that results in photoelectrons maintains the same statistics as the incident light, Fried's work validated that assumption. What we "see" is indeed what is there in the light itself, with the addition of noise added during the propagation from the source to the detector, and during the detection process.

Counting Photons Over Time

Considering photons arriving randomly in time. They are not bunched at one moment or avoiding one another, but simply arriving sporadically. Sampling at a pixel in the image plane selects only photons traveling in a specific direction and among those the arrival times at that pixel are distributed throughout the collection interval. Suppose there are N bins dividing the light along the distance it travels in the collection time, and that p is the probability of finding one photon within one of these bins. If there are n photons distributed randomly in the sample, then for this one measurement the probability of finding one in a chosen bin is

$$p = n \,/\, N \tag{1}$$

When the measurement is repeated many times, always with the same number of bins, the probability must converge to

$$p = \langle n \rangle \, / \, N \tag{2}$$

The probability P of detecting a specific discrete number of random photons n in a window of time or space depends only on the average number in that window taken over many trials. We will see that it is given by the Poisson distribution in the limit of infinitely large N [3]

$$P(n|\langle n \rangle) = \frac{\langle n \rangle^n}{n!} \exp(-\langle n \rangle) \ . \tag{3}$$

Permutations and Combinations of Counted Photons

Pause to recall how many ways we can choose n elements from a set of N [16]. Given the condition that $0 \leq n \leq N$ this is the binomial coefficient

$$\mathcal{C}(N,n) = \binom{N}{n} = \frac{N!}{n!(N-n)!} \tag{4}$$

For example, if $N = 3$ and $n = 2$ we are asking how many ways there are to sort 2 photons into 3 bins. The binomial coefficient for "from 3 choose 2" is 3: $|1|1|0|$, $|0|1|1|$, $|1|0|1|$.

The proof of this expression for the number of combinations uses the number of permutations of N photons taken n at time.

$$\mathcal{P}(N,n) = (N) \times (N-1) \times (N-2) ... \times (N-n+1) = \frac{N!}{(N-n)!} \tag{5}$$

which is analogous but lacks the $n!$ because for permutations the order is important. There are N ways to pick the first one, then $N-1$ possibilities for the second one, $N-2$ for the third and so on down to $N-n+1$ when all the choices are exhausted picking a total of n. Thus for 3 taken 2 at a time we have $\mathcal{P}(3,2) = 3! = 6$. From (A,B,C) that would be (A,B), (B,A), (A,C), (C,A), (B,C), (C,B). The $\mathcal{P}(N,n)$ permutations of N items taken n at a time would be the $\mathcal{C}(N,n)$ combinations of the set, and then ordering those n selections in $\mathcal{P}(n,n)$ ways

$$\mathcal{P}(N,n) = \mathcal{C}(N,n) \cdot \mathcal{P}(n,n) \tag{6}$$

$$\mathcal{C}(N,n) = \frac{N!}{(N-n)!} \cdot \frac{1}{n!} \tag{7}$$

$$\mathcal{C}(N,n) = \frac{N!}{n!(N-n)!} \tag{8}$$

Poisson Distribution of Photons

Insight into the meaning of a Poisson distribution comes from a consideration of the probability of finding n bins with one photon and the remaining $(N-n)$ with none for a finite number of bins N. It should be the same as the probability of a successful detection in n out of N random trials while counting the number in a single bin, that is the binomial distribution

$$P(n|N) = \frac{N!}{n!(N-n)!}\, p^n\, (1-p)^{N-n} \tag{9}$$

where p is the probability of detecting a photon in one trial which we know is $\langle n \rangle / N$. In this expression p^n is the probability that n photons will land in the cell. The term $(1-p)^{N-n}$ is the probability that remainder will *not* land in the cell. The coefficient $N!/n!(N-n)!$ is the relative number of combinations of N possible cells taken n at a time allowing for the necessity of every photon to land in a cell.

Substituting for p we have

$$P(n|N) = \frac{N!}{n!(N-n)!}\, (\frac{\langle n \rangle}{N})^n\, (1-\frac{\langle n \rangle}{N})^{N-n} \tag{10}$$

To assure that there is only one photon in a bin we factor out the N terms and take the limit $N \to \infty$ of

$$P(n|N) = \frac{1}{n!}\frac{N!}{(N-n)!\, N^n}\, \langle n \rangle^n\, (1-\frac{\langle n \rangle}{N})^{N-n} \tag{11}$$

The fraction in the multiplier

$$\frac{1}{n!}\frac{N!}{(N-n)!\, N^n}$$

is found by considering its logarithm

$$\ln(\frac{1}{n!}\frac{N!}{(N-n)!\, N^n})$$

using the Stirling approximation

$$\lim_{N\to\infty} (\ln N!) = N \ln N - N$$

$$\lim_{N\to\infty} \ln(\frac{1}{n!}\frac{N!}{(N-n)!\, N^n}) = 0$$

to show that it is

$$\lim_{N\to\infty} \frac{1}{n!}\frac{N!}{(N-n)!\, N^n} = 1$$

The power term expands

$$(1-\frac{\langle n \rangle}{N})^{N-n} = 1 - (N-n)(\frac{\langle n \rangle}{N})^1 + + \frac{1}{2!}(N-n)(N-n-1)(\frac{\langle n \rangle}{N})^2 + \cdots$$

and taken to the limit becomes

$$1 - \langle n \rangle + \frac{1}{2!}\langle n \rangle^2 - \cdots = \exp(-\langle n \rangle)$$

Combining these two terms in the limit of large N yields the Poisson statistics for detecting random photons

$$P(n|\langle n \rangle) = \frac{\langle n \rangle^n}{n!} \exp(-\langle n \rangle) \tag{12}$$

It depends only on the measured $\langle n \rangle$ which has absorbed the probability of detection p.

Normal Distribution

A normal distribution is a Gaussian of unit area with mean μ, standard deviation σ, and variance σ^2

$$N(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{13}$$

The standard deviation of x is the square root of the mean square deviation of x from μ over the distribution

$$\sigma = \sqrt{\langle (x-\mu)^2 \rangle} \tag{14}$$

One σ from its center the distribution has the value $N(\sigma)/N(0) = e^{-1/2} \approx 0.6065$ of the peak. Farther out, where $N(\gamma)/N(0) = 1/2$, the halfwidth at half maximum (HWHM) γ is given by

$$N(\gamma)/N(0) = \exp\left(-\frac{(\gamma-\mu)^2}{2\sigma^2}\right) = \frac{1}{2} \tag{15}$$

This leads to the HWHM for the normal function that is about 17% more than σ.

$$(\gamma - \mu)^2 = 2\ln 2\, \sigma^2 \tag{16}$$
$$\gamma = \sqrt{2\ln 2}\, \sigma \tag{17}$$
$$\gamma \approx 1.1774\, \sigma \tag{18}$$

The importance of the normal distribution in describing the statistics of light detection lies in its connection with the probability of a value occuring in a random process and the central limit theorem [17]. Where here we have defined $N(x)$ as a Gaussian, in statistical analysis we have a set of values that are randomly distributed. Operationally, that set is described by its mean μ of x

$$\mu = \left(\sum_{i=0}^{i=n} x_i\right)/n \tag{19}$$

and the root-mean-square σ of its deviations from the mean

$$\sigma = \sqrt{\left(\sum_{i=0}^{i=n} (x_i - \mu)^2\right)/n} \tag{20}$$

These two values represent the observable outcome of the random process. As the number of samples $n \to \infty$, the central limit theorem declares that the probability of a value of x in a set of random values approaches the normal distribution for that set which is defined by the mean and standard deviation of the data

$$P(x|\mu, \sigma) \to N(x|\mu, \sigma) \tag{21}$$

The probability of finding a measurement within a range of the mean is found from the error function. It gives an integral of a Gaussian

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)dt \tag{22}$$

From this and how the normal distribution is determined from σ and μ, the probability of a value of x falling within $\pm a$ of μ is

$$P(a|x, \sigma) = \mathrm{erf}(a/\sigma\sqrt{2}) \tag{23}$$

At $a = 1 \times \sigma$ this is 0.6827, at $a = 2 \times \sigma$ it is 0.9545, and at $a = 3 \times \sigma$ it is 0.997. An alternative view is that this is the probablity that a measured mean is the actual value. Thus if we take a mean and standard deviation of many measurements of a fixed quantity where randomness determines the outcome, the probability that the actual value is within 1σ of the mean is 68.3%, while at 3σ it is 99.7%. When there is an expected value from theory or another well-regarded independent measurement that differs from one found in which random error limits the accuracy, at 3σ the differences would very probably be significant.

Detection and Noise

The analysis of the detector statistics led Fried [15] to look closely at how efficiency in the detection process affected the noise in the photoelectron signal, and to conclude it was valid to use the same statistics on the photoelectron signal that we use on the incident photon flux. Here we will ask fundamentally what is the best statistical distribution in practice, determine the properties of the noise in the photon signal, and analyze the signal-to-noise ratio. The central limit theorem invokes a Gaussian distribution to describe the statistics of random processes. There are two for the detection of light, an intrinsic statistical character of the arriving photons, and one imposed on this signal by processes either in the source, the medium between it and the detector, or the detector itself. Indeed, the variance of the Poisson distribution is

$$\sigma_n^2 = \langle (n - \langle n \rangle)^2 P(n|\langle n \rangle) \rangle \tag{24}$$

which we have seen for the Poisson distribution to be

$$\sigma_n^2 = \langle n \rangle \tag{25}$$

The standard deviation σ in n is therefore

$$\sigma_n = \sqrt{\langle n \rangle} \tag{26}$$

This means that if we detect n photons there will be a Poisson distribution about that value and in various trials we will measure a standard deviation of σ_n as intrinsic shot noise in the signal. The resulting ratio of the mean signal to the noise in the signal (SNR) is

$$\mathcal{R}_{SNR} = \langle n \rangle / \sqrt{\langle n \rangle} \tag{27}$$
$$= \sqrt{\langle n \rangle} \tag{28}$$

This is a profoundly useful relationship for observation and measurement of light since it applies to the number of discrete detections, which is to say the efficiency of the detection does not enter. If a sensor only responds to 1 out of 10 photons then the noise in the signal from 10,000 is $\mathcal{R}_{SNR} = \sqrt{10000/10}$, just as if only 1000 photons had been incident. Similarly, a measurement with a noise-to-signal ratio of 1 part in 1000 requires a signal of 1,000,000 detected photons. Typically, the accuracy of a measurement of a faint source in a finite time will be flux-limited.

Non-Poissonian Light

Poisson statistics applies to coherent light which has a perfectly stable flux lasting forever. Light which has more noise is deemed *super-Poissonian* and includes thermal radiation, partially coherent light, and Bose-Einstein statistics of photons. *Sub-Poissonian* light has less noise and would result from photons which are more ordered in time, that is photons that do not have the uniformly random behavior that led to the Poisson distribution [3].

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